Math 53: Multivariable Calculus **Sections 102, 103**

Worksheet for 2020-03-13

Conceptual questions

Problems marked with "**" are more difficult and/or more tangential to the main course content.

Question 1. Consider the graph of $f(x, y)$. Let (a, b) be a point in the plane, and consider the vectors

$$\mathbf{u} = \langle f_x(a,b), f_y(a,b), -1\rangle,$$
$$\mathbf{v} = \langle f_x(a,b), f_y(a,b), |\nabla f(a,b)|^2\rangle.$$

Note that these are vectors in $\mathbb{R}^3$, whose projections into the x, y-plane are equal to the gradient $\nabla f(a, b)$.

Geometrically, how are these vectors related to the graph of f at the point $(a, b, f(a, b))$? (Are they tangent to the graph at that point? Perpendicular? Something else?)

**Can you geometrically interpret the length $|\mathbf{u}|$? (We will return to this question much later in the course.)

Question 2. Is it possible to use Lagrange multipliers for some (nice and differentiable) function $f(x, y)$ on the constraint $x + y = 1$ and find only *one* solution? If so, give an example of such a function f. If not, explain why not.

**Then answer the same question for the constraint $x^2 + y^2 = 1$.

Question 3. Let D be the region in the plane defined by $0 < x^2 + y^2 \leq 1$.

(a) Is this region closed? Is this region bounded?
(b) **Suppose that f is a continuous function on $\mathbb{R}^2$. Show that it has either an absolute maximum or absolute minimum (or maybe both) when constrained to D. Hint: Apply EVT to the region $x^2 + y^2 \leq 1$.
(c) **Find an example of a function f which is continuous on D but does *not* have an absolute maximum or minimum on D. Hint: In view of the preceding part, f cannot be extended to a continuous function defined on $x^2 + y^2 \leq 1$. In other words, the limit $\lim_{(x,y)\to(0,0)} f(x, y)$ must not exist. Use this to help guide your search for a suitable f.

Computations

Problem 1. Find the extrema of the function $f(x, y, z) = x^2 - yz + z$ on the region defined by the two inequalities

$$\begin{cases} x^2 + y^2 + z^2 \leq 4, \\ x^2 + y^2 \geq z. \end{cases}$$

This is the only problem on this worksheet, but it is a very long problem! Here is an overview, so that you can try the problem in advance of Friday's section:

- The overall strategy is to identify all candidates, and then to compare the value of f at all the candidates. The largest will be the absolute max, smallest the absolute min. Of course, this method is only logically sound if we know that f has an absolute max and min to begin with. This is indeed the case—why?
- Ignoring f for the moment, draw a sketch of the constraint region. You will see that it has a 3-dimensional interior part:

$$\begin{cases} x^2 + y^2 + z^2 < 4, \\ x^2 + y^2 > z, \end{cases}$$

two 2-dimensional boundary surfaces:

$$\begin{cases} x^2 + y^2 + z^2 < 4, \\ x^2 + y^2 = z, \end{cases} \qquad \begin{cases} x^2 + y^2 + z^2 = 4, \\ x^2 + y^2 > z, \end{cases}$$

and a 1-dimensional circular edge:

$$\begin{cases} x^2 + y^2 + z^2 = 4, \\ x^2 + y^2 = z. \end{cases}$$

The reason this problem is so long is that we have to find the candidates on each of these parts separately!
- The candidates in the 3-dimensional interior part are just the critical points of f in that region. For the other parts, you have some choice in how to proceed. You could use Lagrange multipliers, try to substitute in an equation to eliminate one of the variables, or parametrization. We will discuss all of these methods on Monday. (time permitting).

Problem 2. This problem is a correction to something I claimed on Friday in the 9AM meeting.

Consider the function $f(x, y) = y - x^2$ constrained to the region $y \geq x^4$. Show that the gradient of f is perpendicular to the constraint boundary at the point $(0, 0)$, but that this point is neither a local maximum nor local minimum of f on this region.

So this shows you that actually determinining whether each candidate is a local extrema is quite subtle. That's why it's a good thing we *don't* actually need to do that when solving absolute extrema problems!